

THE EXPRESSION  
OF  
A QUADRATIC SURD  
AS A  
*CONTINUED FRACTION.*



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### *PREFATORY NOTE.*

IN investigating the properties of a new special form of determinant intimately connected with the subject of continued fractions,\* it was necessary for the author to study somewhat minutely that particular class of continued fractions which are expressible as quadratic surds. The accounts given of this class in the ordinary text-books were found to deal only with the case in which the number under the root-sign is an integer, and to be for the most part fragmentary, and sometimes confusing. On this account it was thought right to publish for the use of students a logically arranged exposition of the subject in its general form, including the more elementary of the new results which had been obtained.

\* The results of the investigation referred to have been communicated to the Royal Society of Edinburgh in a paper entitled "Continuants; a new special class of Determinants."

THE  
 EXPRESSION OF A QUADRATIC SURD  
 AS A  
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1. Every arithmetical expression which is not exactly equal to an integer may be changed into the form of a continued fraction by performing the operations indicated, and finding by the ordinary rule the continued fraction equivalent to the decimal thus resulting.

In particular, any quadratic surd may be treated in this way,

$$\text{e.g. } \sqrt{23} = 4.79583\dots = 4 + \frac{79583}{100000} + \dots$$

$$= 4 + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \dots$$

2. In the case of a quadratic surd, however, it is much easier and much more instructive to proceed *directly* as follows:—

$$\sqrt{\frac{39}{5}} = \frac{\sqrt{195}}{5}$$

$$\begin{array}{r} 5) \sqrt{195} 2 \\ \underline{10} \\ \sqrt{195 - 10} 5( \end{array}$$

or, multiplying divisor and dividend by  $\frac{\sqrt{195 + 10}}{5}$ ,

$$\begin{array}{r} 19) \sqrt{195 + 10} 1 \\ \underline{19} \\ \sqrt{195 - 9} 19( \end{array}$$

or, multiplying divisor and dividend by  $\frac{\sqrt{195+9}}{19}$ ,

$$\begin{array}{r}
 6) \sqrt{195+9}3 \\
 \underline{18} \\
 \sqrt{195-9}6( \\
 \text{or} \qquad \qquad 19) \sqrt{195+9}1 \\
 \qquad \qquad \qquad \underline{19} \\
 \qquad \qquad \qquad \sqrt{195-10}19( \\
 \qquad \qquad \text{or} \qquad \qquad 5) \sqrt{195+10}4 \\
 \qquad \qquad \qquad \qquad \underline{20} \\
 \qquad \qquad \qquad \qquad \sqrt{195-10}5( \\
 \qquad \qquad \qquad \qquad \dots\dots\dots
 \end{array}$$

We do not proceed further, because the last line being exactly the same as the third line of the process, the operations henceforward will continually recur in the cycle already traversed.

$$\therefore \sqrt{\frac{39}{8}} = 2 + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{4} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{4} + \&c.$$

3. The continued fractions thus resulting have necessarily their numerators each equal to unity, and their partial denominators positive integers; and it is only continued fractions of this class which we mean to consider here. It must however be borne in mind that the same quadratic surd can be expressed as a continued fraction with other than unit-numerators, and this too in a variety of ways. For example it is easily shown that  $\sqrt{23}$

$$\text{besides being} = 4 + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \&c.$$

$$\text{also} = 4 + \frac{7}{8} + \frac{7}{8} + \frac{7}{8} + \&c.$$

$$= 3 + \frac{14}{6} + \frac{14}{6} + \frac{14}{6} + \&c.$$

$$= \dots\dots\dots$$

4. No quadratic surd can be expressed as a terminating continued fraction.

For, otherwise, the last convergent being found, we should have a rational fraction equal to the surd, which is impossible.

5. If any positive number which is not an exact square be changed into the form  $\frac{H}{G}$  where  $H$  and  $G$  are integers, the direct process of reduction of  $\sqrt{\frac{H}{G}}$  to a continued fraction with unit numerators may be exhibited as follows:—

1. Suppose  $H > G$  so that  $\sqrt{\frac{H}{G}}$  or  $\frac{\sqrt{HG}}{G} > 1$

$$G) \frac{\sqrt{HG}}{AG} \text{ (A i.e., highest integer in } \frac{\sqrt{HG}}{G}$$

$$\frac{\sqrt{HG} - AG}{G} ($$

or,

$$\begin{array}{l} \text{Call this} \quad H - A^2 G) \sqrt{HG} + AG (Q_1 \\ \text{divisor, } D_1, \quad Q_1 D_1 \quad \text{Write } R_2 \text{ for } Q_1 D_1 - AG \\ \sqrt{HG} - R_2) D_1 ( \end{array}$$

or,

$$\begin{array}{l} \text{Call this} \quad \frac{HG - R_2^2}{D_1} ) \sqrt{HG} + R_2 (Q_2 \\ \text{divisor, } d_2, \quad Q_2 d_2 \quad \text{Write } r_3 \text{ for} \\ \sqrt{HG} - r_3) d_2 ( \quad Q_2 d_2 - R_2 \end{array}$$

or,

$$\begin{array}{l} \text{Call this} \quad \frac{HG - r_3^2}{d_2} ) \sqrt{HG} + r_3 (Q_3 \\ \text{divisor, } d_3, \quad Q_3 d_3 \end{array}$$

the result being

$$\sqrt{\frac{H}{G}} = A + \frac{1}{Q_1} + \frac{1}{Q_2} + \frac{1}{Q_3} + \dots$$

where  $Q_1, Q_2, Q_3, \dots$  are from the nature of the process all positive integers. For clearness in what follows, other capital letters have been used wherever it is perfectly evident that the numbers to be represented are integral.

2. Suppose  $H < G$ , then  $\sqrt{\frac{H}{G}} = \frac{1}{\sqrt{\frac{G}{H}}}$ , and the continued fraction  $= \sqrt{\frac{G}{H}}$  is found in the way shown above, and substituted for it.

6. The process involves an infinite series of divisors, dividends, and quotients. The quotients are obviously *partial quotients*, each with a complete quotient corresponding. Every dividend is the sum of a surd and a rational term, which rational terms may be spoken of shortly as the *rational dividends*.

Clearly, if at any stage we should have simultaneously the same divisor and rational dividend as have before simultaneously occurred, the process may be discontinued, as thenceforward it would be but an unending repetition of what had already been gone through.

7. The divisors  $G, D_1, d_2, d_3, \dots$ , the rational dividends  $0, AG, R_2, r_3, \dots$  and the partial quotients  $A, Q_1, Q_2, Q_3, \dots$  are evidently related to each other in the following manner:—

$$Q_n = \text{highest integer in } \frac{\sqrt{HG + r_n}}{d_n} \dots\dots(1)$$

$$r_{n+1} = Q_n d_n - r_n \dots\dots\dots(2)$$

$$d_{n+1} = \frac{HG - r_{n+1}^2}{d_n} \dots\dots\dots(3)$$

where  $n = 0, 1, 2, 3, \dots$

8. *All the divisors and rational dividends are integral.*

From (2) and (3):—

$$\begin{aligned} d_{n+1} &= \frac{HG - (Q_n d_n - r_n)^2}{d_n} \\ &= \frac{HG - r_n^2}{d_n} - Q_n^2 d_n + 2 Q_n r_n \end{aligned}$$

$$\therefore (3) \quad = d_{n-1} - Q_n^2 d_n + 2 Q_n r_n$$

This equation and (2) show that if  $d_{n-1}, d_n, r_n$  be integers, then  $r_{n+1}, d_{n+1}$  must also be integers. But the first two divisors and second rational dividend are known to be integral, therefore the third divisor and third rational dividend must be integral, and so on.

*N.B.*—Instead of  $D_1, R_2, Q_1, Q_2 \dots$  we shall now for convenience use the corresponding small letters, it being understood throughout what follows that, unless otherwise stated, all numbers represented by letters are integral.

9. *The divisors are all positive.*

Suppose  $d_n$  positive, then—

$$(1) \quad q_n d_n < \sqrt{HG} + r_n$$

$$\therefore (2) \quad \sqrt{HG} > r_{n+1}$$

$$\therefore (3) \quad d_n d_{n+1} > 0$$

So that if any one of the divisors  $d_n$  be positive, so also must the next  $d_{n+1}$ . But the first and second are known to be positive, therefore the third is positive, and so on.

10. *No rational dividend can be greater than the highest integer in the surd-dividend.*

$$(1) \quad q_n d_n < \sqrt{HG} + r_n$$

$$\therefore (3) \quad r_{n+1} + r_n < \sqrt{HG} + r_n$$

$$\therefore \quad r_{n+1} < \sqrt{HG} \dots \dots \dots (4)$$

$$\therefore \quad \overline{\hspace{1cm}} \text{ highest integer in } \sqrt{HG}, \text{ which}$$

integer let us in future indicate by  $I$ .

11. *The rational dividends are all positive.*

Suppose  $r_n$  positive, then—

$$(4) \quad \sqrt{HG} > r_n$$

$$\therefore (1) \quad q_n d_n > r_n$$

$$\therefore (2) \quad r_{n+1} > 0$$

So that if any one of the rational dividends be positive, so also must the next. But the first and second are known to be positive, therefore the third is positive, and so on.

12. *No divisor or partial quotient can be greater than double the highest integer in the surd-dividend.*

For if either  $q_n$  or  $d_n$  were  $> 2I$  then (2) would  $r_n + r_{n+1} > 2I$  contrary to § 10.



13. *A divisor double of the highest integer in the surd-dividend can only occur when the number under the root-sign is less by unity than an exact square.*

For, putting  $d_n = 2I$  in (2) we have  $2Iq_n = r_n + r_{n+1}$

But (§ 10)  $r_n + r_{n+1}$  is not  $> 2I$

$$\therefore q_n = 1$$

$$\text{and } r_n = I = r_{n+1}$$

$$\therefore (3) \quad \frac{HG - I^2}{2I} = d_{n+1} \text{ and } \therefore \overline{\overline{1}}$$

But  $I$  being the highest integer in  $\sqrt{HG}$ ,

$$HG < I^2 + 2I$$

$$\therefore \frac{HG - I^2}{2I} < 1$$

Whence necessarily,  $\frac{HG - I^2}{2I} = 1$

$$\text{and } \therefore HG = (I + 1)^2 - 1$$

14. *If any divisor be subtracted from the next but one, and the rational dividend corresponding to the latter be subtracted from the one preceding it, the first of these remainders is exactly divisible by the second, the quotient being the partial quotient corresponding to the last-mentioned rational dividend.*

$$(3) \quad d_n d_{n+1} = HG - r_{n+1}^2$$

$$\text{and } d_{n-1} d_n = HG - r_n^2$$

$$\therefore (d_{n+1} - d_{n-1}) d_n = r_n^2 - r_{n+1}^2 \\ = (r_n + r_{n+1})(r_n - r_{n+1})$$

$$\text{But (2)} \quad q_n d_n = r_n + r_{n+1}$$

$$\therefore \frac{d_{n+1} - d_{n-1}}{q_n} = r_n - r_{n+1}$$

$$\text{and } \frac{d_{n+1} - d_{n-1}}{r_n - r_{n+1}} = q_n \dots \dots \dots (5)$$

15. *The sum of any divisor and the following rational dividend is greater than the surd-dividend, and the sum of any divisor except the first and the corresponding rational dividend is greater than the surd-dividend.*

From the nature of the process,

$$\begin{aligned} \sqrt{HG} - r_{n+1} &< d_n \\ \therefore d_n + r_{n+1} &> \sqrt{HG} \dots \dots \dots (6) \end{aligned}$$

$$\begin{aligned} (2) \quad r_{n-1} + r_n &= q_{n-1} d_{n-1} \\ \therefore (4) \quad \sqrt{HG} + r_n &> q_{n-1} d_{n-1} \\ &> d_{n-1} \dots \dots \dots (7) \end{aligned}$$

Now, except when  $n=0$  we have from (3)

$$\begin{aligned} d_n &= \frac{HG - r_{n+1}^2}{d_{n-1}} \\ &= \frac{\sqrt{HG} + r_n}{d_{n-1}} (\sqrt{HG} - r_n) \\ \therefore (7) \quad &> \sqrt{HG} - r_n \\ \therefore \text{when } n > 0, d_n + r_n &> \sqrt{HG} \dots \dots \dots (8) \end{aligned}$$

16. *The difference between any two rational dividends except the first is less than either the divisor corresponding to the subtrahend or the divisor preceding that.*

Of any two rational dividends  $r_m, r_n$  let  $r_m$  be the greater.

$$\begin{aligned} (8) \text{ when } n > 0, d_n &> \sqrt{HG} - r_n \\ \text{and } (6) \quad d_{n-1} &> \sqrt{HG} - r_n \\ \text{and } (4) \quad r_m &< \sqrt{HG} \\ \therefore \text{when } n > 0, r_m - r_n &< d_n \text{ and } < d_{n-1} \dots \dots \dots (9) \end{aligned}$$

17. Def<sup>n</sup>. A continued fraction having a set of constituents which recurs uninterruptedly, ad infinitum, is called a *recurring* continued fraction.

In writing such fractions we adopt the notation used in connection with recurring decimals in Arithmetic, only using an asterisk instead of a dot.

Thus  $3 + \frac{1}{1} + \frac{1}{4} + \frac{1}{5} + \frac{1}{2}$  is meant to denote

$$3 + \frac{1}{1} + \frac{1}{4} + \frac{1}{5} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{2} + \frac{1}{4} + \dots \text{ \&c.}$$

The operation of the rule for calculating the numerators and denominators of the convergents of a continued fraction which has unit-numerators, we denote by  $K$ . Thus  $K(a) = a$ ,  $K(a/b)$

$= ab + 1$ ,  $K(a, b, c) = abc + c + a$ , &c., and the convergents of the above continued fraction are  $\frac{K(3, 1)}{K(1)}$ ,  $\frac{K(3, 1, 4)}{K(1, 4)}$ ,  $\frac{K(3, 1, 4, 5)}{K(1, 4, 5)}$  &c.

18. *The continued fraction with unit-numerators and positive integral partial-denominators, found as an equivalent of any quadratic surd, is always a recurring continued fraction, the end of the cycle being reached before more than  $2 I^2$  partial quotients have been found.*

For no divisor can be greater than  $2 I$  (§ 12) and no rational dividend can be greater than  $I$  (§ 10); so that by the time that  $2 I \times I$  divisions have been performed every possible divisor must have occurred along with every possible rational dividend; and consequently, if a divisor and rational dividend have not already for the second time appeared simultaneously—thus necessitating a cycle in the operations of dividing—such must take place before more than  $2 I^2$  divisions have been performed.

19. *In every case the cycle begins with the second operation of division.*

Suppose that the  $(s+1)^{\text{th}}$  operation of division is exactly the same as a previous one, the  $(e+1)^{\text{th}}$  say, so that

$$d_s = d_e, r_s = r_e \text{ and } \therefore q_s = q_e$$

$$\text{From (3) } d_{s-1} d_s = HG - r_s^2 \text{ and } d_{e-1} d_e = HG - r_e^2$$

$$\therefore (\text{hyp.}) \quad d_{s-1} = d_{e-1} \dots \dots \dots (10)$$

$$\text{Again (2) } r_{s-1} + r_s = q_{s-1} d_{s-1} \text{ and } r_{e-1} + r_e = q_{e-1} d_{e-1}$$

$\therefore$  by subtraction we have (10) and (hyp.)

$$(r_{s-1} \sim r_{e-1}) \div d_{e-1} = q_{s-1} \sim q_{e-1}$$

But (§ 16) when  $e > 1$

$$r_{s-1} \sim r_{e-1} < d_{e-1}$$

The only possible result therefore is

$$e \text{ being } > 1, r_{s-1} = r_{e-1} \text{ and } q_{s-1} = q_{e-1}$$

$$e \text{ being } = 1, r_{e-1} = 0, \text{ consequently } r_{s-1} \text{ is not } = r_{e-1} \text{ and } q_{s-1} \\ \text{is not } = q_{e-1}$$

We have thus shown that some one of the operations of division beyond the second being known to recur, its predecessor must also recur; and going backwards in this manner we see that in

every case the *second* operation of division must recur. Further, the second operation being known to recur we have proved that the first cannot, therefore the cycle commences with the second operation.

Thus, up to this point we have learned that every quadratic surd when expressed as a continued fraction with unit-numerators takes the form—

$$A + \frac{1}{\underset{*}{q_1} + \frac{1}{\underset{*}{q_2} + \frac{1}{\underset{*}{q_3} + \dots + \frac{1}{\underset{*}{q_i} + \dots}}}$$

or

$$\frac{1}{\underset{*}{A} + \frac{1}{\underset{*}{q_1} + \frac{1}{\underset{*}{q_2} + \frac{1}{\underset{*}{q_3} + \dots + \frac{1}{\underset{*}{q_i} + \dots}}}}$$

according as the surd is greater or less than unity.

20. Defn. The first partial quotient of all may thus be spoken of as the *unique partial quotient*.

21. The last member of the cycle of partial quotients is double of the unique partial quotient; the last member of the cycle of divisors is equal to the first divisor of all; and the last member of the cycle of rational dividends is equal to the first member of the cycle.

From (3)  $d_{i+1} d_i = HG - r_{i+1}^2$

But (§ 19)  $d_{i+1} = d_1$  and  $r_{i+1} = r_1$  and  $\therefore = AG$

$\therefore d_1 d_i = HG - A^2 G^2 = d_1 G$  (§ 5)

$\therefore d_i = G$  i.e.  $d_0$

Again,  $q_{i-1} d_{i-1} = r_i + r_{i-1} = 2r_i + r_{i-1} - r_i$

$\therefore$  (§ 16)  $< 2r_i + d_i$

$\therefore d_{i-1} < 2r_i + G$

and  $\frac{d_{i-1}}{G} < 2 \frac{r_i}{G} + 1 \dots \dots \dots (11)$

Now (3)  $Gd_{i-1} = HG - r_i^2$ ,  $r_i$  being thus a multiple of  $G$ ,

$\therefore \left(\frac{r_i}{G}\right)^2 + \frac{d_{i-1}}{G} = \frac{H}{G}$

But from (11) we see that the highest integer in the square

root of the first member is  $\frac{r_i}{G}$  and the highest integer in  $\sqrt{\frac{H}{G}}$  is of course  $A$ .

$$\begin{aligned} \therefore \quad r_i &= A G \text{ i.e. } r_1. \\ \text{Lastly, since (2)} \quad q_i d_i &= r_i + r_{i+1} \\ \therefore \quad q_i G &= A G + A G \\ \therefore \quad q_i &= 2 A. \end{aligned}$$

22. *If the last member of the cycle of partial quotients be omitted, what remains is the same when read backwards as when read forwards; the same is true of the cycle of divisors; and of the cycle of rational dividends when the last member is included. That is, if  $l$  be the number of members in the cycle, then*

$$r_i = r_1 \text{ (already proved)}$$

$$d_{l-1} = d_1 \quad q_{l-1} = q_1 \quad r_{l-1} = r_2$$

$$d_{l-2} = d_2 \quad q_{l-2} = q_2 \quad r_{l-2} = r_3$$

$$\dots\dots\dots$$

$$\begin{aligned} \text{For (3)} \quad d_i d_{l-1} &= HG - r_i^2 \\ \therefore (\S 21) \quad G d_{l-1} &= HG - A^2 G^2 \\ \therefore \quad d_{l-1} &= H - A^2 G = d_1 \quad (\S 5) \end{aligned}$$

$$\begin{aligned} \text{From (2)} \quad q_{l-1} &= (r_{l-1} + r_i) \div d_{l-1} \\ \therefore (\S 21) \quad &= (r_{l-1} + r_1) \div d_{l-1} \end{aligned}$$

$$\text{But} \quad q_1 = (r_1 + r_2) \div d_1$$

$$\therefore \quad q_{l-1} \sim q_1 = (r_{l-1} \sim r_2) \div d_{l-1} \text{ (or } d_1)$$

$$\text{and } (\S 16) \quad r_{l-1} \sim r_2 < d_{l-1} \text{ or } d_1$$

$$\therefore \quad q_{l-1} = q_1 \text{ and } r_{l-1} = r_2$$

If we now suppose that

$$d_{l-h} = d_h, \quad q_{l-h} = q_h \text{ and } r_{l-h} = r_{h+1} \dots\dots\dots (12)$$

it is evident that we may, by the above method, prove

$$d_{l-(h+1)} = d_{h+1}, \quad q_{l-(h+1)} = q_{h+1} \text{ and } r_{l-(h+1)} = r_{h+2}$$

But (12) is true when  $h = 1$   $\therefore$  it must also be true when  $h = 2$ , and so on.

Thus, every quadratic surd when expressed as a continued

fraction with unit-numerators takes the form

$$A + \underbrace{\frac{1}{q_1}}_* + \frac{1}{q_2} + \dots + \frac{1}{q_2} + \frac{1}{q_1} + \underbrace{\frac{1}{2A}}_* + \dots$$

or its reciprocal.

23. All the terms of the cycle of partial quotients except the last constitute what may be called the *symmetric portion* of the cycle, and by the *middle term* or *terms* of the cycle is meant the middle term or terms of the symmetric portion of the cycle: similarly in the case of the cycle of divisors.

24. *In the cycle of rational dividends no two consecutive terms can be equal without being the middle terms of the cycle; and the divisor and partial quotient corresponding to the former of the two must each be the middle term of its own cycle.*

For suppose  $r_{n+1} = r_n \dots \dots \dots (13)$

$$\text{then } (3) \left. \begin{array}{l} d_{n+1} d_n = HG - r_{n+1}^2 \\ \text{and } d_n d_{n-1} = HG - r_n^2 \end{array} \right\}$$

$$\therefore (13) \quad d_{n+1} = d_{n-1} \dots \dots \dots (14)$$

$$\text{From } (2) \left. \begin{array}{l} q_{n+1} d_{n+1} = r_{n+1} + r_{n+2} \\ \text{and } q_{n-1} d_{n-1} = r_{n-1} + r_n \end{array} \right\}$$

$$\therefore (13) \text{ and } (14) \quad q_{n+1} \sim q_{n-1} = (r_{n+2} \sim r_{n-1}) \div d_{n-1} \text{ (or } d_{n+1})$$

$$(\S 16) \quad < 1$$

$$\therefore q_{n+1} = q_{n-1} \text{ and } r_{n+2} = r_{n-1} \dots \dots \dots (15)$$

Again from (14) and (15) we may show that

$$r_{n+3} = r_{n-2}$$

and so on. Therefore  $r_{n+1}$  and  $r_n$  are the middle terms of their cycle. As in the two other cycles one of the terms (§ 23) is not reckoned in finding the middle term, it is clear that the middle term of each is that corresponding to  $r_n$ .

25. The preceding theorem is a result of practical importance.

Ex. Expand  $\sqrt{\frac{17}{12}}$  as a continued fraction with unit-numerators.

$$\begin{array}{r}
12) \sqrt{204}(1 \\
\quad 12 \\
\hline
\sqrt{204-12}12( \\
\quad 5) \sqrt{204+12}(5 \\
\quad \quad 25 \\
\hline
\sqrt{204-13}5( \\
\quad 7) \sqrt{204+13}(3 \\
\quad \quad 21 \\
\hline
\sqrt{204-8}7( \\
\quad 20) \sqrt{204+8}(1 \\
\quad \quad 20 \\
\hline
\sqrt{204-12}20( \\
\quad 3) \sqrt{204+12}(8 \\
\quad \quad 24 \\
\hline
\sqrt{204-12} \\
\hline
\text{.....}
\end{array}$$

Observing here that  $r_5 = 12 = r_4$ , we proceed no further, since we know that  $q_4$ , i.e. 8 must be the middle term of the cycle of partial quotients. We have thus

$$\sqrt{\frac{17}{12}} = 1 + \frac{1}{5} + \frac{1}{3} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{2} + \dots$$

\* \*

26. *If two consecutive divisors be alike and also the two corresponding partial quotients, they must be the middle terms of their respective cycles.*

For suppose

$$d_n = d_{n+1} \text{ and } q_n = q_{n+1}$$

$$\text{then } q_n d_n = q_{n+1} d_{n+1}$$

$$\therefore (2)$$

$$r_n + r_{n+1} = r_{n+1} + r_{n+2}$$

$$\therefore r_n = r_{n+2} \dots \dots \dots (16)$$

$$\text{Hence } (3) \quad d_n d_{n-1} = d_{n+2} d_{n+1}$$

$$\therefore (\text{hyp.}) \quad d_{n-1} = d_{n+2} \dots \dots \dots (17)$$

$$\left. \begin{array}{l}
\text{But } (2) \quad q_{n-1} d_{n-1} = r_{n-1} + r_n \\
\text{and } q_{n+2} d_{n+2} = r_{n+2} + r_{n+3}
\end{array} \right\}$$

$$\therefore (16) \text{ and } (17) \quad q_{n-1} \sim q_{n+3} = (r_{n-1} \sim r_{n+3}) \div d_{n+2} \text{ (or } d_{n-1})$$

$$(\S 16) \quad < 1$$

$$\therefore q_{n-1} = q_{n+3} \text{ and } r_{n-1} = r_{n+3}$$

Again, from these results we may show that

$$d_{n-2} = d_{n+3} \text{ and } q_{n-2} = q_{n+3}$$

and so on. Therefore  $d_n$  and  $d_{n+1}$ ,  $q_n$  and  $q_{n+1}$  are the middle terms of their respective cycles.

27. The following is an illustrative example of the practical value of the preceding theorem.

Ex. Expand  $\sqrt{\frac{1}{61}}$  as a continued fraction with unit-numerators.

$$\begin{array}{r}
 1) \sqrt{61}(7 \\
 \quad 7 \\
 \hline
 \sqrt{61-7}1( \\
 \quad 12) \sqrt{61+7}(1 \\
 \quad \quad 12 \\
 \hline
 \quad \sqrt{61-5}12( \\
 \quad \quad 3) \sqrt{61+5}(4 \\
 \quad \quad \quad 12 \\
 \hline
 \quad \quad \sqrt{61-7}3( \\
 \quad \quad \quad 4) \sqrt{61+7}(3 \\
 \quad \quad \quad \quad 12 \\
 \hline
 \quad \quad \quad \sqrt{61-5}4( \\
 \quad \quad \quad \quad 9) \sqrt{61+5}(1 \\
 \quad \quad \quad \quad \quad 9 \\
 \hline
 \quad \quad \quad \quad \sqrt{61-4}9( \\
 \quad \quad \quad \quad \quad 5) \sqrt{61+4}(2 \\
 \quad \quad \quad \quad \quad \quad 10 \\
 \hline
 \quad \quad \quad \quad \quad \sqrt{61-6}5( \\
 \quad \quad \quad \quad \quad \quad 5) \sqrt{61+6}(2 \\
 \quad \quad \quad \quad \quad \quad \quad \dots\dots\dots
 \end{array}$$

Here we have two consecutive partial quotients equal and also the two corresponding divisors equal, and we proceed no further,



knowing that these must be the middle terms of their respective cycles. The result  $\therefore$  is

$$\sqrt{\frac{1}{61}} = \frac{1}{7} + \frac{1}{1} + \frac{1}{4} + \frac{1}{3} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{1} + \frac{1}{14} + \dots$$

\* \*

28. Since  $\frac{K(A, q_1 \dots q_n)}{K(q_1 \dots q_n)}$  is more or less approximately  $= \sqrt{\frac{H}{G}}$ , it is clear that  $GK(A, q_1 \dots q_n)^2 - HK(q_1 \dots q_n)^2$  must be a comparatively small quantity; and, similarly, since  $\frac{K(A, q_1 \dots q_{n-1})}{K(q_1 \dots q_{n-1})}$  and  $\frac{K(A, q_1 \dots q_n)}{K(q_1 \dots q_n)}$  are approximations to  $\sqrt{\frac{H}{G}}$  and their product therefore approximately equal to  $\frac{H}{G}$ , it follows that  $GK(A, q_1 \dots q_{n-1})K(A, q_1 \dots q_n) - HK(q_1 \dots q_{n-1})K(q_1 \dots q_n)$  must be comparatively small. These considerations serve as a preparation for, and an introduction to, two theorems which elegantly express any divisor and rational dividend in terms of the preceding partial quotients and the given number.

$$\begin{aligned} 29. \quad (-1)^{n+1} d_{n+1} &= GK(A, q_1 \dots q_n)^2 - HK(q_1 \dots q_n)^2 \\ (-1)^n r_{n+1} &= GK(A, q_1 \dots q_n)K(A, q_1 \dots q_{n-1}) - HK(q_1 \dots q_n)K(q_1 \dots q_{n-1}) \end{aligned}$$

If at any stage in the calculation of the convergent we use the *complete* quotient instead of the corresponding partial quotient the result is not a *convergent* to the given surd, but an expression exactly *equal* to it. Using, then, the  $(n+1)^{\text{th}}$  complete quotient, we have

$$\frac{\frac{\sqrt{HG} + r_{n+1}}{d_{n+1}} K(A, q_1 \dots q_n) + K(A, q_1 \dots q_{n-1})}{\frac{\sqrt{HG} + r_{n+1}}{d_{n+1}} K(q_1 \dots q_n) + K(q_1 \dots q_{n-1})} = \frac{\sqrt{HG}}{G}.$$

Clearing of fractions and equating the rational terms of the one side to those of the other, and the irrational to the irrational, there results

$$\left. \begin{aligned} r_{n+1} K(A, q_1 \dots q_n) + d_{n+1} K(A, q_1 \dots q_{n-1}) &= HK(q_1 \dots q_n) \dots (18) \\ r_{n+1} K(q_1 \dots q_n) + d_{n+1} K(q_1 \dots q_{n-1}) &= GK(A, q_1 \dots q_n) \dots (19) \end{aligned} \right\}$$

Eliminating from these equations  $r_{n+1}$  and  $d_{n+1}$  in succession, we have

$$d_{n+1} = \frac{HK(q_1 \dots q_n)^2 - GK(A, q_1 \dots q_n)^2}{K(A, q_1 \dots q_{n-1})K(q_1 \dots q_n) - K(A, q_1 \dots q_n)K(q_1 \dots q_{n-1})}$$

$$\therefore (-1)^{n+1} d_{n+1} = GK(A, q_1 \dots q_n)^2 - HK(q_1 \dots q_n)^2 \dots \dots \dots (20)$$

$$\text{and } (-1)^n r_{n+1} = GK(A, q_1 \dots q_{n-1}) - HK(q_1 \dots q_n)K(q_1 \dots q_{n-1}) \dots (21)$$

The most important particular case of these equations is that in which  $d_{n+1}$  is the last divisor of the cycle. (18), (19) and (20) then become

$$AGK(A, q_1 \dots q_1) + GK(A, q_1 \dots q_2) = HK(q_1 \dots q_1) \dots \dots \dots (22)$$

$$AK(q_1 \dots q_1) + K(q_1 \dots q_2) = K(A, q_1 \dots q_1) \dots \dots \dots (23)$$

$$GK(A, q_1 \dots q_1)^2 + HK(q_1 \dots q_1)^2 = \pm G \dots \dots \dots (24)$$

the last of which is notable as affording a solution in positive integers of the indeterminate equation  $x^2 - Ny^2 = 1$  where  $N$  is any given positive integer or fraction.

30.  $\frac{K(A, q_1 \dots q_m)^2}{K(q_1 \dots q_m)^2}$  and  $\frac{K(A, q_1 \dots q_n)^2}{K(q_1 \dots q_n)^2}$  being each approximately equal to  $\frac{H}{G}$ , so also must  $\frac{K(A, q_1 \dots q_m)^2 + K(A, q_1 \dots q_n)^2}{K(q_1 \dots q_m)^2 + K(q_1 \dots q_n)^2}$  which in magnitude lies between them. For particular values of  $m$  and  $n$  this approximation is perfect as we shall now show.

31. In the case of a cycle of partial quotients which consists of  $2z+1$  terms  $\frac{K(A \dots q_{z-s-1})^2 + K(A \dots q_{z+s})^2}{K(q_1 \dots q_{z-s-1})^2 + K(q_1 \dots q_{z+s})^2} = \frac{H}{G}$ , where  $s$  is zero or any integer less than  $z$ .

For, there being  $2z+1$  terms in the cycles,  $d_z = d_{z+1}$ ,  $d_{z-1} = d_{z+2}$  and generally  $d_{z-s} = d_{z+s+1}$

$$\text{and } \therefore (-1)^{z-s} d_{z-s} = -(-1)^{z+s+1} d_{z+s+1}$$

therefore (§ 29.)

$$GK(A \dots q_{z-s-1})^2 - HK(q_1 \dots q_{z-s-1})^2 = HK(q_1 \dots q_{z+s})^2 - GK(A \dots q_{z+s})^2$$

whence the theorem.

There are two particular cases worthy of notice, viz. by taking  $s=0$  and  $z-1$ .

$$\frac{K(A \dots q_{z-1})^2 + K(A \dots q_z)^2}{K(q_1 \dots q_{z-1})^2 + K(q_1 \dots q_z)^2} = \frac{H}{G} \dots \dots \dots (25)$$

$$\frac{A^2 + K(A \dots q_{z-1})^2}{1 + K(q_1 \dots q_{z-1})^2} = \frac{H}{G} \dots \dots \dots (26)$$

32. In the case of a cycle of partial quotients which consists of  $2z$  terms

$$\frac{K(A \dots q_{z-s-2}) K(A \dots q_{z-s-1}) + K(A \dots q_{z+s-1}) K(A \dots q_{z+s})}{K(q_1 \dots q_{z-s-2}) K(q_1 \dots q_{z-s-1}) + K(q_1 \dots q_{z+s-1}) K(q_1 \dots q_{z+s})} = \frac{H}{G}$$

where  $s$  is zero or any integer less than  $z-1$ .

For there being  $2z$  terms in the cycles it follows that  $r_z = r_{z-1}$ ,  $r_{z-1} = r_{z+2}$  and generally  $r_{z-s} = r_{z+s+1}$

$$\text{and } \therefore (-1)^{z-s-1} r_{z-s} = -(-1)^{z+s} r_{z+s+1}$$

$$\begin{aligned} \therefore (\S 29) GK(A \dots q_{z-s-2}) K(A \dots q_{z-s-1}) - HK(q_1 \dots q_{z-s-2}) K(q_1 \dots q_{z-s-1}) \\ = -GK(A \dots q_{z+s-1}) K(A \dots q_{z+s}) + HK(q_1 \dots q_{z+s-1}) K(q_1 \dots q_{z+s}) \end{aligned}$$

whence the theorem.

An important particular case is that in which  $s=0$ , viz.:-

$$\frac{K(A \dots q_{z-1}) \{K(A \dots q_{z-2}) + K(A \dots q_z)\}}{K(q_1 \dots q_{z-1}) \{K(q_1 \dots q_{z-2}) + K(q_1 \dots q_z)\}} = \frac{H}{G} \dots \dots \dots (27)$$

33. If instead of using the last partial quotient in calculating in the ordinary way the convergent corresponding to it we use its half, the numerator of the result divided by the preceding denominator is exactly equal to the given number whose square root was to be found.

First, when the number of terms in the cycle is odd,  $2z+1$ , we know (25) that

$$\frac{H}{G} = \frac{K(A \dots q_{z-1})^2 + K(A \dots q_z)^2}{K(q_1 \dots q_{z-1})^2 + K(q_1 \dots q_z)^2}$$

$$\therefore (\text{App.}) = \frac{K(A, q_1 \dots q_z, q_z \dots q_1, A)}{K(q_1 \dots q_z, q_z \dots q_1)} \text{ as was to be proved.}$$

Second, when the number of terms in the cycle is even,  $2z$ , we know (27) that

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$$\begin{aligned}\frac{H}{G} &= \frac{K(A \dots q_z) K(A \dots q_{z-1}) + K(A \dots q_{z-1}) K(A \dots q_{z-2})}{K(q_1 \dots q_z) K(q_1 \dots q_{z-1}) + K(q_1 \dots q_{z-1}) K(q_1 \dots q_{z-2})}, \quad \therefore (\text{App.}) \\ &= \frac{K(A, q_1 \dots q_{z-1}, q_z, q_{z-1} \dots q_1, A)}{K(q_1 \dots q_{z-1}, q_z, q_{z-1} \dots q_1)}, \quad \text{as was to be proved.}\end{aligned}$$

34. In the case of a cycle of partial quotients which consists of  $2z$  terms

$$\frac{K(A \dots q_z)^2 - K(A \dots q_{z-2})^2}{K(q_1 \dots q_z)^2 - K(q_1 \dots q_{z-2})^2} = \frac{H}{G}.$$

$$\text{For } q_z K(A \dots q_{z-1}, q_z, q_{z-1} \dots A) = K(A \dots q_z)^2 - K(A \dots q_{z-2})^2$$

$$\text{and } q_z K(q_1 \dots q_{z-1}, q_z, q_{z-1} \dots q_1) = K(q_1 \dots q_z)^2 - K(q_1 \dots q_{z-2})^2$$

and thus the theorem follows from the preceding.

35. The fact (§ 18) that the partial quotients recur in a cycle might lead us to expect a simplification of some kind in the calculation of the convergents, and the further fact (§ 22) of the peculiar nature of the cycle induces the same expectation. The way in which this is realized will be explained in what immediately follows.

36. If the cycle of partial quotients consist of  $2n$  terms and the first  $n+1$  convergents have been calculated, the  $2n^{\text{th}}$  convergent may be found without calculating the intermediate ones.

$$\text{Let } \sqrt{\frac{H}{G}} = A + \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_n} + \dots + \frac{1}{q_2} + \frac{1}{q_1} + \frac{1}{2A} + \dots$$

\* \*

then the  $2n^{\text{th}}$  convergent is

$$\frac{K(A, q_1, \dots, q_n, \dots, q_1)}{K(q_1, \dots, q_n, \dots, q_1)}$$

$$\text{But this} = \frac{K(A, q_1 \dots q_n) K(q_1 \dots q_{n-1}) + K(A, q_1 \dots q_{n-1}) K(q_1 \dots q_{n-2})}{K(q_1 \dots q_{n-1}) \{K(q_1 \dots q_n) + K(q_1 \dots q_{n-2})\}}$$

Now if the first  $n+1$  convergents have been calculated,  $K(A, q_1 \dots q_n)$ ,  $K(A, q_1 \dots q_{n-1})$ ,  $K(q_1 \dots q_n)$ ,  $K(q_1 \dots q_{n-1})$ ,  $K(q_1 \dots q_{n-2})$  are all known, and thus the theorem is proved.

37. If the cycle of partial quotients consist of  $2n+1$  terms and the first  $n+1$  convergents have been calculated, the  $(2n+1)^{\text{th}}$  convergent may be found without calculating the intermediate ones.

$$\text{Let } \sqrt{\frac{H}{G}} = A + \underbrace{\frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_n} + \frac{1}{q_n} + \dots + \frac{1}{q_2} + \frac{1}{q_1} + \frac{1}{2A}}_{*} + \dots$$

then the  $(2n+1)^{\text{th}}$  convergent is

$$\frac{K(A, q_1 \dots q_n, q_n \dots q_1)}{\frac{K(q_1 \dots q_n, q_n \dots q_1)}{K(q_1 \dots q_n)^2 + K(q_n \dots q_1)^2}}$$

which =  $\frac{K(A, q_1 \dots q_n) K(q_1 \dots q_n) + K(A, q_1 \dots q_{n-1}) K(q_1 \dots q_{n-1})}{K(q_1 \dots q_n)^2 + K(q_n \dots q_1)^2}$

and is thus dependent upon convergents preceding the  $(n+2)^{\text{th}}$ .

38. If the cycle of partial quotients consist of  $z$  terms and the  $mz^{\text{th}}$  convergent has been calculated, the  $2mz^{\text{th}}$  convergent may be found without calculating the intermediate ones.

Let the cycle of  $z$  terms be  $q_1, q_2, \dots, q_z, q_1, 2A$  then the  $2z^{\text{th}}$  convergent is

$$\begin{aligned} & \frac{K(A, q_1 \dots q_1, 2A, q_1 \dots q_1)}{K(q_1 \dots q_1, 2A, q_1 \dots q_1)} \text{ which} \\ &= \frac{K(A, q_1 \dots q_1, 2A) K(q_1 \dots q_1) + K(A, q_1 \dots q_1) K(q_1 \dots q_2)}{K(q_1 \dots q_1) \{K(q_1 \dots q_1, 2A) + K(q_1 \dots q_2)\}} \\ &= \frac{\{2AK(A \dots q_1) + K(A \dots q_2)\} K(q_1 \dots q_1) + K(A \dots q_1) K(q_1 \dots q_2)}{K(q_1 \dots q_1) \{2AK(q_1 \dots q_1) + 2K(q_1 \dots q_2)\}} \\ &= \frac{\{AK(A \dots q_1) + K(A \dots q_2)\} K(q_1 \dots q_1) + K(A \dots q_1) \{AK(q_1 \dots q_1) + K(q_1 \dots q_2)\}}{2K(q_1 \dots q_1) K(A, q_1 \dots q_1)} \\ &= \frac{HK(q_1 \dots q_1)^2 + GK(A \dots q_1)^2}{2GK(q_1 \dots q_1) K(A, q_1 \dots q_1)} \text{ by (22)} \end{aligned}$$

The theorem is thus true when  $m=1$ , and it is clear that the mode of proof holds for all values of  $m$ . (Observe that a convergent found in this way is not in its simplest form,  $G$  being a factor of both numerator and denominator.)

39. From these theorems we see that in no case is it necessary to continue the ordinary method of calculating the convergents after the middle of the cycle of partial quotients has been reached.

#### EXAMPLES.

$$\text{Ex. 1. } \sqrt{\frac{18}{7}} = 1 + \underbrace{\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{10} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2}}_{*} + \dots$$

Here the first *six* convergents are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{85}{53}$$

$$\begin{aligned} \therefore (\S 36) \text{ the } \textit{tenth} \text{ convergent} &= \frac{85 \times 5 + 8 \times 3}{5(3 + 53)} \\ &= \frac{449}{280} \end{aligned}$$

$$\begin{aligned} \therefore (\S 38) \text{ the } \textit{twentieth} &= \frac{18(280)^2 + 7(449)^2}{2.7.280.449} \\ &= \frac{403201}{251440} \end{aligned}$$

$$\begin{aligned} \dots\dots\dots \text{the } \textit{fortieth} &= \frac{18(251440)^2 + 7(403201)^2}{2.7.251440.403201} \end{aligned}$$

and so on.

$$\text{Ex. 2. } \sqrt{\frac{346}{25}} = 3 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{6} + \dots$$

\* \*

Here the first *four* convergents are

$$\frac{3}{1}, \frac{4}{1}, \frac{11}{3}, \frac{15}{4},$$

$$\begin{aligned} \therefore (\S 37) \text{ the } \textit{seventh} \text{ convergent} &= \frac{15 \times 4 + 11 \times 3}{3^2 + 4^2} \\ &= \frac{93}{25} \end{aligned}$$

$$\begin{aligned} \therefore (\S 38) \text{ the } \textit{fourteenth} &= \frac{346(25)^2 + 25(93)^2}{2.25.25.93} \\ &= \frac{17299}{4650} \end{aligned}$$

whence, in a similar manner, the *twenty-eighth*; and so on.

40. We have shown that a quadratic surd can always be expressed as an interminate continued fraction with unit-numerators and positive integral partial-denominators, that this continued fraction is a recurring one, that the cycle commences with the second partial quotient, that the last member of the cycle is double the first partial quotient of all, and that, if this last member be omitted, the cycle is the same when read backwards as when read

forwards. The converse of this, viz.: that any continued fraction of this nature must be equal to a quadratic surd, is established as a case of the following remarkable theorem.

41. If  $A, q_1, q_2 \dots$  be any finite quantities whatever, positive or negative, rational or irrational, &c., then

$$A + \frac{1}{\underset{*}{q_1} + \frac{1}{q_2 + \dots + \frac{1}{q_2 + \underset{*}{q_1} + \frac{1}{2A + \dots}}}} = \sqrt{\frac{K(A, q_1 \dots q_1, A)}{K(q_1 \dots q_1)}}$$

$$\text{For suppose } A + \frac{1}{\underset{*}{q_1} + \frac{1}{q_2 + \dots + \frac{1}{q_2 + \underset{*}{q_1} + \frac{1}{2A + \dots}}}} = x$$

$$\text{then } A + \frac{1}{q_1 + \frac{1}{q_2 + \dots + \frac{1}{q_1 + \frac{1}{2A + x - A}}}} = x.$$

$$\text{i.e. } \frac{K(A, q_1 \dots q_1, A + x)}{K(q_1 \dots q_1, A + x)} = x$$

$$\therefore \frac{K(A, q_1 \dots q_1, A) + xK(A, q_1 \dots q_1)}{K(A, q_1 \dots q_1) + xK(q_1 \dots q_1)} = x$$

Clearing of fractions and simplifying, we have

$$K(A, q_1 \dots q_1, A) = x^2 K(q_1 \dots q_1)$$

$$\therefore x = \sqrt{\frac{K(A, q_1 \dots q_1, A)}{K(q_1 \dots q_1)}}$$

42. We thus see that if a continued fraction of the kind we have been considering be given we are able at once to find the quadratic surd equivalent to it.

#### EXAMPLES.

Ex. 1. Find the quadratic surds which are equal to

$$2 + \frac{1}{\underset{*}{1} + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \dots}}}}} \text{ and } \frac{1}{3 + \frac{1}{1 + \frac{1}{5 + \frac{1}{1 + \frac{1}{6 + \dots}}}}}$$

$$2 + \frac{1}{\underset{*}{1} + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \dots}}}}} = \sqrt{\frac{K(2, 1, 1, 1, 1, 2)}{K(1, 1, 1, 1)}} \\ = \sqrt{\frac{34}{5}}$$

$$\frac{1}{3 + \frac{1}{\underset{*}{1} + \frac{1}{5 + \frac{1}{1 + \frac{1}{6 + \dots}}}}} = \text{reciprocal of } 3 + \frac{1}{\underset{*}{1} + \frac{1}{5 + \frac{1}{1 + \frac{1}{6 + \dots}}}}$$

$$\begin{aligned}
&= \sqrt{\frac{K(1, 5, 1)}{K(3, 1, 5, 1, 3)}} \\
&= \sqrt{\frac{7}{104}}
\end{aligned}$$

Ex. 2. Find the general expression for all the quadratic surds which when expressed as continued fractions with unit-numerators have 1, 1, 1, 1, for the symmetric portion of the cycle of partial quotients.

$$\begin{aligned}
n + \frac{1}{\underset{*}{1}} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{\underset{*}{2n} + \dots} &= \sqrt{\frac{K(n, 1, 1, 1, 1, n)}{K(1, 1, 1, 1)}} \\
&= \sqrt{\frac{5n^2 + 6n + 2}{5}}
\end{aligned}$$

which is thus the general expression wanted. Putting  $n=2$  we have the particular example given above.

The wide generality of the theorem should not however, be forgotten.

#### EXAMPLE.

$$\begin{aligned}
\sqrt[3]{2} + \frac{1}{\underset{*}{\sqrt[3]{-16}}} + \frac{1}{0} + \frac{1}{\sqrt[3]{-16}} + \frac{1}{2\sqrt[3]{2} + \dots} \\
&= \sqrt{\frac{K(\sqrt[3]{2}, \sqrt[3]{-16}, 0, \sqrt[3]{-16}, \sqrt[3]{2})}{K(\sqrt[3]{-16}, 0, \sqrt[3]{-16})}} \\
&= \sqrt{\frac{8\sqrt[3]{-1} + 2\sqrt[3]{2}}{4\sqrt[3]{-2}}} \\
&= \sqrt{(\sqrt[3]{4} + \frac{1}{2}\sqrt[3]{-1})}
\end{aligned}$$

#### SURDS OF THE FORM $\sqrt{H}$ , $H$ BEING AN INTEGER.

43. Of course the preceding theorems all hold in reference to such quadratic surds as  $\sqrt{17}$ ,  $\sqrt{5}$ , &c., where the extrahend (i.e., the number under the root-sign) is an integer. It has only to be remembered that in this case  $G=1$ , the *surd-dividend* is the same as the *given surd*, and the *highest integer in the surd-dividend* ( $I$ ) is the same as the *unique partial quotient* ( $A$ ), so that the expression of the theorems may be somewhat simplified. We now proceed to theorems referring to the special case only.



44. *When the extrahend is integral, a term equal to the unique partial quotient may occur in the cycle of partial quotients, but only in one position, viz.: as middle term.*

For suppose  $q_n = A$

then, (2)  $A d_n = r_n + r_{n+1}$

$\therefore$  (§ 10)  $d_n$  is not  $> 2$ .

Now it is proved incidentally in § 21, that if  $d_n = G$ , then  $q_n = 2A$ , consequently  $d_n$  here cannot = 1 (*i.e.*,  $G$ ) for (hyp.)  $q_n = A$ . Of necessity, then,

$$d_n = 2$$

and  $\therefore$  (§ 10)  $r_n = r_{n+1} = A$

so that (§ 24)  $q_n$  must be the middle term of the cycle.

#### EXAMPLES.

$$\sqrt{31} = 5 + \frac{1}{1} + \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{3} + \frac{1}{1} + \frac{1}{1} + \frac{1}{10} + \dots$$

\* \*

$$\sqrt{131} = 11 + \frac{1}{2} + \frac{1}{4} + \frac{1}{11} + \frac{1}{4} + \frac{1}{2} + \frac{1}{22} + \dots$$

\* \*

$$\sqrt{(3n+1)^2 + 4n + 2} = (3n+1) + \frac{1}{1} + \frac{1}{2} + \frac{1}{3n+1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2(3n+1)} + \dots$$

\* \*

Here 5, 11,  $3n+1$ , the highest integers in the given surds are found to occur in the respective cycles of partial quotients, but only as middle terms.

45. *Although, when the extrahend is integral, the cycle of partial quotients may contain a term equal to the unique partial quotient (§ 44), and must contain one double of the same (§ 21), it cannot contain a term lying in value between these.*

For (2)  $q_n d_n = r_n + r_{n+1} \leq 2A$  (§ 10)

so that if  $q_n > A$ ,  $d_n$  must = 1, and consequently (§ 21)  $q_n = 2A$ . That is, every partial quotient greater than  $A$  must be equal to  $2A$ .

46. When the extrahend is integral and there are  $2z$  terms in the cycle of partial quotients,  $K(A \dots q_{z-2}) + K(A \dots q_z)$  is exactly divisible by  $K(q_1 \dots q_{z-1})$ , and further if the middle term be odd  $K(A \dots q_{z-1})$  is exactly divisible by  $K(q_1 \dots q_{z-2}) + K(q_1 \dots q_z)$ .

This is at once seen to follow from (27) § 32, if it be borne in mind that  $K(q_1 \dots q_{z-1})$  is prime to  $K(A \dots q_{z-1})$  and, when  $q_z$  is odd,  $K(q_1 \dots q_{z-2}) + K(q_1 \dots q_z)$  prime to  $K(A \dots q_{z-2}) + K(A \dots q_z)$ .

47. When the extrahend (H) is a prime and there are  $2z$  terms in the cycle of partial quotients, then

$$\begin{aligned} K(A \dots q_{z-1}) &= K(q_1 \dots q_{z-2}) + K(q_1 \dots q_z) \\ K(A \dots q_{z-2}) + K(A \dots q_z) &= H K(q_1 \dots q_{z-1}) \end{aligned}$$

and the middle term is  $A$  or  $A - 1$  according as  $A$  is odd or even.

The two identities and the fact that the middle term is odd may be demonstrated from (27) § 32. Supposing this done we have

$$\begin{aligned} K(A \dots q_{z-1}) &= K(q_1 \dots q_{z-2}) + K(q_1 \dots q_z) \\ \therefore A K(q_1 \dots q_{z-1}) + K(q_2 \dots q_{z-1}) &= 2 K(q_1 \dots q_{z-2}) + q_z K(q_1 \dots q_{z-1}) \\ \text{or } (A - q_z) K(q_1 \dots q_{z-1}) + K(q_2 \dots q_{z-1}) &= 2 K(q_1 \dots q_{z-2}) \end{aligned}$$

Now, since  $K(q_1 \dots q_{s-1}) > K(q_1 \dots q_{s-2})$ , it follows that  $A - q_s$  is not  $> 1$ , that is,  $q_s$  is not  $< A - 1$ ; and we know (§ 45) that it is not  $> A$ , therefore, since it is odd, it must be  $A$  or  $A - 1$  according as  $A$  is odd or even.

### EXAMPLES.

$$\sqrt{47} = 6 + \underbrace{\frac{1}{1}}_* + \frac{1}{5} + \frac{1}{1} + \underbrace{\frac{1}{12}}_* + \dots$$

$$\sqrt{107} = 10 + \underbrace{\frac{1}{2}}_* + \frac{1}{1} + \frac{1}{9} + \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{20}}_* + \dots$$

$$\sqrt{239} = 15 + \underbrace{\frac{1}{2} + \frac{1}{5} + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{15} + \frac{1}{4} + \frac{1}{2} + \frac{1}{1} + \frac{1}{5} + \frac{1}{2}}_{*} + \underbrace{\frac{1}{30}}_{*} + \dots$$

48. When the extrahend (H) is integral and the middle term ( $q_z$ ) of the cycle of partial quotients is equal to the unique partial quotient, then

$$K(q_2 \dots q_{z-1}) = 2 K(q_1 \dots q_{z-2})$$

$$\text{and } A^2 + \frac{2 K(q_2 \dots q_z)}{K(q_1 \dots q_{z-1})} = H.$$

From (27) § 32  $\frac{2 K(A \dots q_{z-1})}{K(q_1 \dots q_z) + K(q_1 \dots q_{z-2})} = \text{an integer,}$

$$\text{i.e. } \frac{2 A K(q_1 \dots q_{z-1}) + 2 K(q_2 \dots q_{z-1})}{q_z K(q_1 \dots q_{z-1}) + 2 K(q_1 \dots q_{z-2})} = \dots\dots\dots$$

But  $q_z = A \quad \therefore K(q_2 \dots q_{z-1}) = 2 K(q_1 \dots q_{z-2})$

Further, we have at the same time,

$$K(A \dots q_{z-1}) = K(q_1 \dots q_z) + K(q_1 \dots q_{z-2}) \quad \text{and } \therefore (27)$$

$$H K(q_1 \dots q_{z-1}) = K(A \dots q_{z-2}) + K(A \dots q_z)$$

$$= 2 A K(q_1 \dots q_{z-2}) + K(q_2 \dots q_{z-2}) + A^2 K(q_1 \dots q_{z-1}) + K(q_2 \dots q_z)$$

$$= A K(q_2 \dots q_{z-1}) + K(q_2 \dots q_{z-2}) + A^2 K(q_1 \dots q_{z-1}) + K(q_2 \dots q_z)$$

$$= K(q_2 \dots q_z) + A^2 K(q_1 \dots q_{z-1}) + K(q_2 \dots q_z)$$

$$\therefore H = A^2 + \frac{2 K(q_2 \dots q_z)}{K(q_1 \dots q_{z-1})}.$$

49. By the second of the two identities just proved, there is suggested a theorem which may be derived from § 41, viz. :—

$$A + \frac{1}{q_1} + \dots + \frac{1}{q_{z-1}} + \frac{1}{A} + \frac{1}{q_{z-1}} + \dots + \frac{1}{q_1} + \frac{1}{2A} + \dots$$

\* \*

$$= \sqrt{\left\{ A + \frac{2 K(q_2 \dots q_z)}{K(q_1 \dots q_{z-1})} \right\}}$$

50. Further,  $K(q_2 \dots q_{z-1})$  being even, it follows that  $K(q_2 \dots q_z)$  and  $K(q_1 \dots q_{z-1})$ , and therefore also  $K(q_2 \dots q_z) \div K(q_1 \dots q_{z-1})$  must be odd; and thus we observe, from the same identity, that when the square root of an integer has the middle term of its cycle of partial quotients equal to the unique partial quotient, the excess of the integer above the highest square in it is of the form  $4s + 2$ .

51. Any integer (H), whose square root when expressed as a continued fraction with unit-numerators has an odd number of terms in its cycle of partial quotients, must be the sum of two integral squares prime to each other.

Let the number of terms in the cycle be  $2z + 1$ , then

$$d_z = d_{z+1}$$

and  $\therefore$  (3)  $H - r_{z+1}^2 = d_{z+1}^2$

so that  $H = d_{z+1}^2 + r_{z+1}^2$

and we can show by a *reductio ad absurdum* that  $d_{z+1}$  and  $r_{z+1}$  are mutually prime.

### EXAMPLES.

$$\sqrt{13} = 3 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{6} + \dots \quad \text{and } 13 = 2^3 + 3^2$$

$$\sqrt{41} = 6 + \frac{1}{\underset{*}{2}} + \frac{1}{2} + \frac{1}{\underset{*}{12}} + \dots \quad \text{and } 41 = 4^2 + 5^2$$

$$\sqrt{365} = 19 + \frac{1}{9} + \frac{1}{1} + \frac{1}{9} + \frac{1}{38} + \dots \quad \text{and } 365 = 11^2 + 12^2$$

52. The converse of the preceding theorem is not universally true; thus  $34 = 3^2 + 5^2$ , while

$$\sqrt{34} = 5 + \underbrace{\frac{1}{1}}_* + \frac{1}{4} + \frac{1}{1} + \underbrace{\frac{1}{10}}_* + \dots$$

53. The general expression for every integer whose square root, when expressed as a continued fraction with unit-numerators has  $q_1, q_2, \dots, q_n, q_1$  for the symmetric portion of its cycle of partial denominators, is

$$\{\frac{1}{2}K(q_1..q_1)M - (-1)^{l_1}\frac{1}{2}K(q_1..q_2)K(q_2..q_2)\}^2 + K(q_1..q_2)M - (-1)^{l_1}K(q_2..q_2)^2$$

1 being the number of terms in the cycle.

The general expression for *every* such number, fractional as well as integral, is (§ 41)

$$\frac{K(A, q_1, q_2 \dots q_2, q_1, A)}{K(q_1, q_2 \dots q_2, q_1)} \dots\dots\dots (a)$$

so that we have only to determine what form for  $A$  is necessary and sufficient to make this expression integral. Now

$$\begin{aligned} K(A, q_1 \dots q_1, A) &= AK(q_1 \dots q_1, A) + K(q_2 \dots q_2, A) \\ &= A^2 K(q_1 \dots q_1) + 2AK(q_1 \dots q_2) + K(q_2 \dots q_2) \\ \therefore \frac{K(A, q_1 \dots q_1, A)}{K(q_1 \dots q_1)} &= A^2 + \frac{2AK(q_1 \dots q_2) + K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \dots (28) \end{aligned}$$

so that it is necessary and sufficient that

$$\frac{2AK(q_1 \dots q_2) + K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \text{ be integral,}$$

$$\therefore \text{ that } \frac{2AK(q_1 \dots q_2)^2 + K(q_1 \dots q_2)K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \dots\dots\dots,$$

$$\text{i.e. that } \frac{2A\{K(q_1 \dots q_1)K(q_2 \dots q_2) + (-1)^2\} + K(q_1 \dots q_2)K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \dots\dots\dots,$$

$$\therefore \text{ that } \frac{(-1)^2 2A + K(q_1 \dots q_2)K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \dots\dots\dots,$$

$$\text{or that } A = \frac{1}{2} K(q_1 \dots q_1) M - (-1)^2 \frac{1}{2} K(q_1 \dots q_2) K(q_2 \dots q_2)$$

where  $M$  is zero or any integer.

Making this substitution for  $A$  in the alternative form of (a) given by (28) the second term becomes

$$\frac{K(q_1 \dots q_2)K(q_1 \dots q_1)M - (-1)^2 K(q_1 \dots q_2)^2 K(q_2 \dots q_2) + K(q_2 \dots q_2)}{K(q_1 \dots q_1)}$$

$$\text{or } K(q_1 \dots q_2)M - (-1)^2 \frac{K(q_2 \dots q_2) \{K(q_1 \dots q_2)^2 - (-1)^2\}}{K(q_1 \dots q_1)}$$

$$\text{or } K(q_1 \dots q_2)M - (-1)^2 K(q_2 \dots q_2)^2$$

and adding the other term we have the required expression

$$\{\frac{1}{2} K(q_1 \dots q_1)M - (-1)^2 \frac{1}{2} K(q_1 \dots q_2)K(q_2 \dots q_2)\}^2 + K(q_1 \dots q_2)M - (-1)^2 K(q_2 \dots q_2)$$

54. Ex. Find all the integers whose square roots when expressed as continued fractions have each 1, 1, 2, 1, 1, for the symmetric portion of its cycle of partial denominators.

$$\begin{aligned}\text{Here } K(q_1 \dots q_1) &= K(1, 1, 2, 1, 1) = 12 \\ K(q_1 \dots q_2) &= K(1, 1, 2, 1) = 7 \\ K(q_2 \dots q_2) &= K(1, 2, 1) = 4\end{aligned}$$

and therefore the general expression for the required integers is, by the preceding theorem,

$$(6M - 14)^2 + 7M - 16.$$

Giving  $M$  the values 3, 4, 5, ..... this becomes 21, 112, 275, ..... and thus we know that

$$\begin{aligned}\sqrt{21} &= 4 + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{8} + \dots \\ &\quad * \qquad \qquad \qquad * \\ \sqrt{112} &= 10 + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{20} + \dots \\ &\quad * \qquad \qquad \qquad * \\ \sqrt{275} &= 16 + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{32} + \dots \\ &\quad * \qquad \qquad \qquad *\end{aligned}$$

and so on.

#### OTHER EXAMPLES.

$$\begin{aligned}\sqrt{(2M+1)^2 + 4} &= (2M+1) + \frac{1}{M} + \frac{1}{1} + \frac{1}{1} + \frac{1}{M} + \frac{1}{4M+2} + \dots \\ &\quad * \qquad \qquad \qquad * \\ \sqrt{4M^2 + 4M - 3} &= 2M + \frac{1}{1} + \frac{1}{M-1} + \frac{1}{2} + \frac{1}{M-1} + \frac{1}{1} + \frac{1}{4M} + \dots \\ &\quad * \qquad \qquad \qquad *\end{aligned}$$

55. No integer can be found whose square root when expressed as a continued fraction with unit-numerators has  $q_1, q_2, \dots, q_2, q_1$  for the symmetric portion of its cycle of partial denominators, unless either  $K(q_1 \dots q_2)$  or  $K(q_2 \dots q_2)$  be even.

If both were odd we see that the general expression in § 53 would be fractional unless  $K(q_1 \dots q_1)$  were also odd, and this is impossible, for then

$$K(q_1 \dots q_2) + K(q_2 \dots q_2) \text{ which } = K(1, q_1 \dots q_2)$$

$$\text{and } K(q_1 \dots q_1) + K(q_1 \dots q_2) \text{ which } = K(1, q_1 \dots q_1)$$

would both be even, whereas they are seen to be mutually prime

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#### HISTORICAL NOTE.

THERE is one very important fact in the history of the subject which has not yet found its way into text-books and books of reference—viz., that the square root of an integer was expressed as a continued fraction (*e.g.*  $\sqrt{18} = 4 + \frac{2}{8} + \frac{2}{8} + \frac{2}{8} + \dots$ ) by Cataldi in his *Trattato del modo brevissimo di trovare la radice quadrata delli numeri*, published in 1613, seven years before the birth of Lord Brounker, the commonly reputed introducer of continued fractions. (V. Libri, *Histoire des sciences Mathématiques en Italie* T. IV., p. 87-98.)